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# AUTOMORPHISM GROUPS AND THE FULL STATE SPACES OF THE PETERSEN GRAPH GENERALIZATIONS OF $G_{32}$*

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The geometric duals of the generalized Petersen graphs $G(n, k)$ are the Greechie representations of the Generalizations of $G_{32}$. The duals are denotes by $G^*(n, k)$ and the generalizations by $L(G^*(n, k))$. For these generalizations which are orthomodular posets and lattices, the automorphism groups are completely determined. State properties are also investigated with the following results obtaining.

(1) $L(G^*(n, 1))$ admits a full set of dispersion free states if $n$ is even.
(2) $L(G^*(n, 1))$ does not admit a full set of states if $n$ is odd.
(3) $L(G^*(n, 2))$ admits a full set of dispersion free states for all values of $n$ other than 5 or 8.
(4) $L(G^*(8, 2))$ admits a full set of states but does not admit a full set of dispersion free states.
(5) $L(G^*(5, 2))$ does not admit a full set of states.
(6) $L(G^*(n, 3))$ admits a full set of dispersion free states for all $n$.

The geometric duals of the generalized Petersen graphs $G(n, k)$ are the Greechie representations of the generalizations of $G_{32}$. These generalizations are denoted by $L(G^*(n, k))$. In Section 1 necessary and sufficient conditions are given for determining which of these generalizations are orthomodular posets and which are orthomodular lattices. The automorphism groups are then completely determined using results from graph theory and a previous paper.

In Section 2 the relations between full-state properties of $L(G^*(n, k))$ and certain edge numbering properties of $G(n, k)$ are given. These are used to answer completely the question of existence of full sets of states and full sets of dispersion free states for all $L(G^*(n, k))$ which are orthomodular posets or lattices when $k = 1$, 2, and 3.

## 1. The generalizations of $G_{32}$ and their automorphism groups

Throughout this paper, results and terminology from the theory of graphs, orthomodular posets (OMPS), and orthomodular lattices (OMLS), will be used. Readers unfamiliar with these concepts are referred to [5] and [6].

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An *incidence structure* is a triple $S = (P, B, I)$ where $P$, $B$, and $I$ are sets with $P \cap B = \emptyset$ and $I \subseteq P \times B$. The elements of $P$ are called *points* and those of $B$ called *blocks*. If $(p, b) \in I$, we say $p$ and $b$ are incident. We usually write $pIb$ or $bIp$.

The *dual structure* $S^* = (P^*, B^*, I^*)$ of $S$ is defined by $P^* = B$, $B^* = P$, and $(b, p) \in I^*$ if and only if $(p, b) \in I$.

We note that a graph $G$ is an incidence structure with $P$ as the set of points and $B$ as the set of lines and with incidence defined in the usual way. We refer to $G^*$ as the geometric dual of $G$ and use $L(G^*)$ to denote the orthostructure whose Greechie diagram is $G^*$ (see also [3] and [7] for further details).

Throughout this paper, we consider only those orthostructures which are formed by pastings of distributive lattices of the form $2^n$. These distributive lattices then become maximal Boolean sublattices and are referred to as blocks. The Greenchie diagrams of the orthostructures so formed are essentially a top view of those orthostructures with only the atoms showing as points and the blocks appearing as straight lines or smooth arcs passing through the atoms of these blocks. (See [6, pp. 44 ff.] for a more detailed description of these Greechie diagram representations.) Note that these representations are incidence structures with the atoms as points and the maximal Boolean sublattices as blocks.

If one takes the dual structure of the Greechie diagram $G_{32}$, the well-known Petersen graph is obtained. Figure 1 shows this as well as the original Greechie diagram of $G_{32}$.

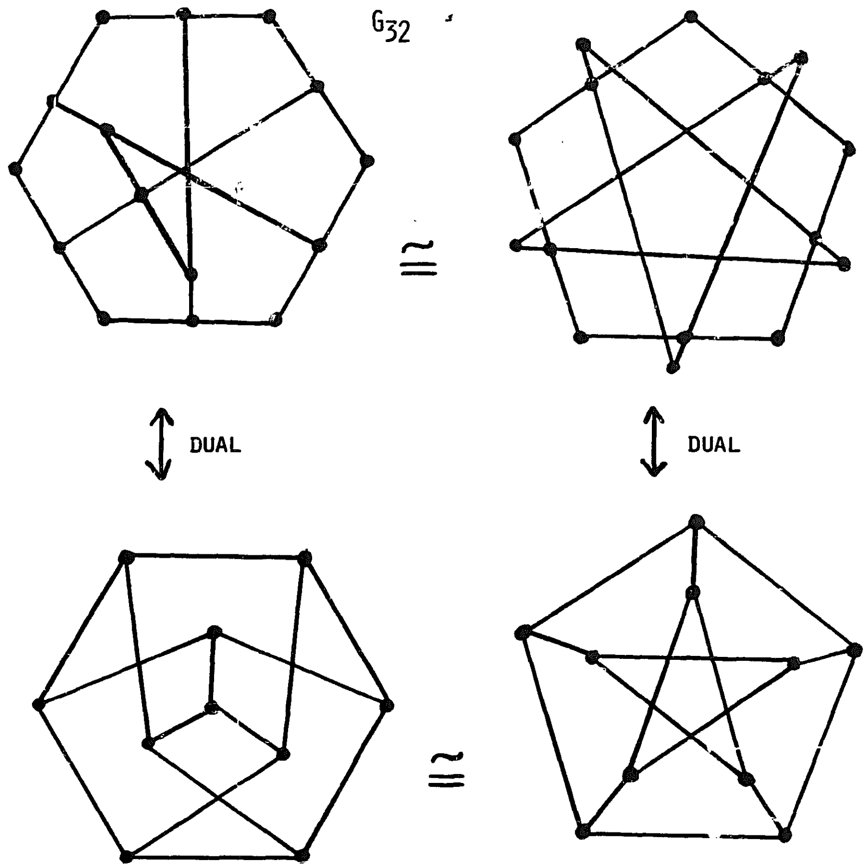
The generalizations of the Petersen graph $G(n, k)$ as defined by Watkins [8] and Coxeter [2] are as follows. The points are $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}$ and the edges are $[u_i, u_{i+1}]$, $[v_i, v_{i+k}]$, and $[u_i, v_i]$ where all subscripts are modulo $n$ and $1 < 2k < n$. These will be denoted $a_i$, $b_i$, and $c_i$ respectively. Note that the Petersen graph is $G(5, 2)$. The polygon generated by $u_0, u_1, \ldots, u_{n-1}$ is called the *outer rim* of $G(n, k)$. Each connected component of the subgraph generated by $v_0, v_1, \ldots, v_{n-1}$ is called an *inner rim*. The edges $[u_i, v_i]$ are called *spokes*. If two lines $x$ and $y$ are adjacent we will write $x \operatorname{adj} y$. Otherwise we write $x \not\operatorname{adj} y$. The following is noted in [8].

**Remark 1.** $G(n, k)$ has precisely $(n, k)$ inner rims, each of which has length $n/(n, k)$, where $(n, k)$ denotes the greatest common divisor of $n$ and $k$.

Combining the above remark with the well-known Atomistic Loop Lemma of Greechie [4] and results in [7], we have the following

**Lemma 2.** *$L(G^*(n, k))$ is an OMP (respectively OML) if only if $n/(n, k) \neq 3$ and $n \neq 3$ (respectively, $k \neq 1$ and $n(n, k) \neq 3$ or 4 and $n \neq 3$ or 4).*

As in [7] we see that $\operatorname{Aut}(G(n, k)) \cong \operatorname{Aut}(G^*(n, k)) \cong \operatorname{Aut}(L(G^*(n, k)))$. Combining the above results with those of Watkins, Graver and Frucht [9], we have the following




THE PETERSEN GRAPH

Fig. 1.

**Theorem 3.** *Let $G(n, k)$ be such that $L(G^*(n, k))$ is an OMP and let $S_n$ denote the symmetric group on $n$ elements. Then*

(1) $\mathrm{Aut}(L(G^*(4, 1))) \cong S_4 \times S_2$,

$\mathrm{Aut}(L(G^*(5, 2))) \cong S_5$,

$\mathrm{Aut}(L(G^*(8, 3))) \cong \langle \rho, \delta, \sigma \rangle$ *where* $\rho^8 = \delta^2 = \sigma^3 = I$, $\delta\rho\delta = \rho^{-1}$, $\delta\sigma\delta = \sigma^{-1}$, $\sigma\rho\sigma = \rho^{-1}$ *and* $\sigma\rho^4 = \rho^4\sigma$.

$\mathrm{Aut}(L(G^*(10, 2))) \cong \langle \rho, \sigma \rangle$, *where* $\rho^{10} = \sigma^3 = (\sigma\rho^2)^2 = \rho^5\sigma\rho^{-5}\sigma^{-1} = I$.

$\mathrm{Aut}(L(G^*(10, 3))) \cong S_5 \times S_2$.

$\mathrm{Aut}(L(G^*(12, 5))) \cong \langle \rho, \delta, \sigma \rangle$, *where* $\rho^{12} = \delta^2 = \sigma^3 = I$, $\delta\rho\delta = \rho^{-1}$, $\delta\sigma\delta = \sigma^{-1}$, $\sigma\rho\sigma = \rho^{-1}$ *and* $\sigma\rho^4 = \rho^4\sigma$.

$\mathrm{Aut}(L(G^*(24, 5))) \cong \langle \rho, \sigma \rangle$, *where* $(\sigma\rho)^2 = \sigma^3 = I$ *and* $\rho^4\sigma = \sigma\rho^4$.

*Now suppose $L(G^*(n, k))$ is none of the above.*

(2) *If* $k^2 \neq \pm 1 \pmod n$, *then* $\mathrm{Aut}(L(G^*(n,k))) \cong \langle \rho, \delta \rangle$, *where* $\rho^n = \delta^2 = I$ *and* $\delta\rho\delta = \rho^{-1}$. *This is the dihedral group* $D_n$.

(3) *If* $k^2 = 1 \pmod n$, *then* $\mathrm{Aut}(L(G^*(n,k))) \cong \langle \rho, \delta, \sigma \rangle$, *where* $\rho^n = \delta^2 = \sigma^2 = I$, $\delta\rho\delta = \rho^{-1}$, $\sigma\delta = \delta\sigma$, *and* $\sigma\rho\sigma = \rho^k$.

(4) *If* $k^2 = -1 \pmod n$, *then* $\mathrm{Aut}(L(G^*(n,k))) \cong \langle \rho, \sigma \rangle$, *where* $\rho^n = \sigma^4 = I$ *and* $\sigma\rho\sigma^{-1} = \rho^k$.

*In every case,* $D_n$ *is a subgroup of* $\mathrm{Aut}(L(G^*(n,k)))$.

## 2. States on the generalizations of $G_{32}$

A *state* on an ortholattice or orthomodular poset $L$ is defined as a map of $L$ into the real unit interval $[0, 1]$ such that $\alpha(1) = 1$ and $\alpha(a_0 \vee a_1) = \alpha(a_0) + \alpha(a_1)$ for $a_0 \perp a_1$. It is *deterministic* or *dispersion free* if $\alpha(x) \in \{0, 1\}$ for all $x \in L$. A set $M$ of states is full if $a \leqslant b$ is equivalent to $\alpha(a) \leqslant \alpha(b)$ for $\alpha$ in $M$. Note that because of the orthomodular identity $\alpha(a) \leqslant \alpha(b)$ if $a \leqslant b$ is always true in an OMP.

Since $L(G^*(5,2)) = G_{32}$ and since $G_{32}$ does not admit a full set of states [1] it is natural to wonder which of the generalizations of $G_{32}$ that are OMPS or OMLS admit a full set of states. This question will be answered in this section for $k = 1$, 2, and 3.

**Lemma 4.** *A set of states S on an orthomodular poset L is full if and only if the following condition holds:* $x, y \in L$ *and* $x \not\perp y$ *imply there exists* $\alpha \in S$ *such that* $\alpha(x) + \alpha(y) > 1$.

**Proof.** Suppose there exist $x, y$ such that $x \not\perp y$ but $\alpha(x) + \alpha(y) \leqslant 1$. $\alpha(x) + \alpha(y) \leqslant 1 = \alpha(y \vee y') = \alpha(y) + \alpha(y')$ since $y \perp y'$. Thus $\alpha(x) \leqslant \alpha(y')$. However, $x \nleqslant y'$ since $x \not\perp y$. Thus $S$ is not full.

Conversely suppose $S$ is not full but the condition holds. Then there exist $x, y$ such that $\alpha(x) \leqslant \alpha(y)$ but $x \nleqslant y$. This says that $x \not\perp y'$ so that by the condition, $\alpha(x) + \alpha(y') > 1 = \alpha(y) + \alpha(y')$. Thus $\alpha(x) > \alpha(y)$. This is a contradiction. □

**Lemma 5.** *Let* $G(n,k)$ *be such that* $L(G^*(n,k))$ *is an OMP. Then* $L(G^*(n,k))$ *admits a full set of dispersion free states if and only if there exists a set E of edge numberings consisting of zeroes and ones on* $G(n,k)$ *with the following properties:*

(1) *Only one of the edges incident with a given point is numbered with a one.*

(2) *For any two edges which are not adjacent there exists an element of E such that the given edges are numbered with a one.*

*Furthermore,* $L(G^*(n,k))$ *admits a full set of states if and only if there exists a set F of edge numberings consisting of real numbers between zero and one inclusive on* $G(n,k)$ *with the following properties:*

(1′) *The sum of the numberings on adjacent edges is one.*

(2′) *For any edges which are not adjacent there exists an element of F such that the sum of the numberings of the given edges is greater than one.*

**Proof.** First observe that the condition $\alpha(x)+\alpha(y)>1$ in Lemma 4 can be replaced by $\alpha(x)=\alpha(y)=1$ if the states are dispersion free. To define a state on $L(G^*(n, k))$ it suffices to define the state values on the atoms. Finally observe that two distinct atoms are orthogonal if and only if they lie in the same block and that points in $G(n, k)$ represent blocks in $L(G^*(n, k))$ and lines in $G(n, k)$ represent atoms in $L(G^*(n, k))$. The result now follows from Lemma 4. □

**Theorem 6.** *Let $G(n, 1)$ be such that $L(G^*(n, 1))$ is an OMP. Then $L(G^*(n, 1))$ admits a full set of dispersion free states if n is even. Conversely, if n is odd, then $L(G^*(n, 1))$ does not admit a full set of states.*

**Proof.** First note that $n\geq 3$ since $1<2k<n$.

In all edge numberings which follow note that Condition 1 of Lemma 5 is satisfied.

Suppose that $n$ is even. Before proceeding with individual cases, for each $n=4, 6, 8, \ldots$ we define $n$ edge numberings on $G(n, 1)$ as follows: For each $i=0, 1, \ldots, n-1$ define

$$\begin{aligned}\alpha_i(a_i)&=\alpha_i(b_i)=\alpha_i(c_{i+2})\\&=\alpha_i(c_{i+3})=\cdots=\alpha_i(c_{i+n-1})=1 \quad\text{and}\quad \alpha_i(x)=0\end{aligned}$$

for all other edges. It can be shown that if $a_s \not\text{adj}\, c_t$, then there exists $\alpha_i$ such that $\alpha_i(a_s)=\alpha_i(c_t)=1$. It can also be shown that if $b_s \not\text{adj}\, c_t$ then there exists $\alpha_i$ such that $\alpha_i(b_s)=\alpha_i(c_t)=1$. If $n\geq 6$ it is also true that there exists $\alpha_i$ such that $\alpha_i(c_s)=\alpha_i(c_t)=1$. (Note that $c_s \not\text{adj}\, c_t$ is always true for $s\neq t$.)

$n=4$. In addition to the $\alpha_i$'s defined above, we define the 9 edge numberings on $G(4, 1)$ as follows. (From now on we will indicate only those edges which are numbered with a one with all other edges being understood to be numbered with zeroes.)

For $i=0, 1, 2, 3$ define $\beta_i$ by $\beta_i(a_i)=\beta_i(b_i)=\beta_i(a_{i+2})=\beta_i(b_{i+2})=1$.

Define $\gamma_i$ by $\gamma_i(a_i)=\gamma_i(a_{i+2})=\gamma_i(b_{i+1})=\gamma_i(b_{i+3})=1$.

Define $\delta$ by $\delta(c_i)=1$.

It can be shown that Condition 2 of Lemma 5 is satisfied by the $\beta_i$'s whenever $a_s \not\text{adj}\, b_s$, $a_s \not\text{adj}\, b_{s+2}$, $a_s \not\text{adj}\, a_{s+2}$, and $b_s \not\text{adj}\, b_{s+2}$. Condition 2 is satisfied by the $\gamma_i$'s for $a_s \not\text{adj}\, b_{s+1}$ and $a_s \not\text{adj}\, b_{s+3}$. Finally, Condition 2 is satisfied by $\delta$ for $c_s \not\text{adj}\, c_t$. This shows that $L(G^*(4, 1))$ admits a full set of dispersion free states.

$n=6$. In addition to the $\alpha_i$'s defined above, we define the 18 edge numberings on $G(6, 1)$ as follows. For each $i=0, 1, \ldots, 5$ let $\beta_i(a_i)=\beta_i(a_{i+3})=\beta_i(b_i)=\beta_i(b_{i+3})=\beta_i(c_{i+2})=\beta_i(c_{i+5})=1$. Let $\gamma_i(a_{i+2k})=\gamma_i(b_{i+2k+1})=1$ for all $k=0, 1, 2$. Finally, let $\delta_i(a_{i+2k})=\delta_i(b_{i+2k})=1$ for all $k=0, 1, 2$.

Condition 2 of Lemma 5 is satisfied by the $\beta_i$'s for $a_s \mathfrak{adj}\, a_{s+3}$, $b_s \mathfrak{adj}\, b_{s+3}$, $a_s \mathfrak{adj}\, b_s$, $a_s \mathfrak{adj}\, b_{s+3}$.

Condition 2 is satisfied by the $\gamma_i$'s for $a_s \mathfrak{adj}\, a_{s+2}$, $a_s \mathfrak{adj}\, a_{s+4}$, $b_s \mathfrak{adj}\, b_{s+2}$, $b_s \mathfrak{adj}\, b_{s+4}$, $a_s \mathfrak{adj}\, b_{s+1}$, and $a_s \mathfrak{adj}\, b_{s+5}$. Finally, Condition 2 is satisfied by the $\delta_i$'s for $a_s \mathfrak{adj}\, b_{s+2}$, and $a_s \mathfrak{adj}\, b_{s+4}$. Thus, $L(G^*(6, 1))$ admits a full set of dispersion free states.

$n = 8$. In addition to the $\alpha_i$'s, define the 24 edge numberings on $G(8, 1)$ as follows. For each $i = 0, 1, \ldots, 7$ let $\beta_i(a_{i+3k}) = \beta_i(b_{i+3k}) = 1$ for $k = 0, 1, 2$. Also let $\beta_i(c_{i+2}) = \beta_i(c_{i+5}) = 1$. Let $\gamma_i(a_{i+2k}) = \gamma_i(b_{i+2k+1}) = 1$ for $k = 0, 1, 2, 3$. Finally, let $\delta_i(a_{i+2k}) = \delta_i(b_{i+2k}) = 1$ for $k = 0, 1, 2, 3$.

Condition 2 of Lemma 5 is satisfied by the $\beta_i$'s for $a_s \mathfrak{adj}\, a_t$ except $t = s + 4$; $b_s \mathfrak{adj}\, b_t$ except $t = s + 4$; and $a_s \mathfrak{adj}\, b_t$ except $t = s + 1$, $t = s + 4$, and $t = s + 7$.

Condition 2 is satisfied by the $\gamma_i$'s for $a_s \mathfrak{adj}\, b_{s+1}$ and $a_s \mathfrak{adj}\, b_{s+7}$. It is satisfied by the $\delta_i$'s for $a_s \mathfrak{adj}\, a_{s+4}$, $a_s \mathfrak{adj}\, b_{s+4}$, and $b_s \mathfrak{adj}\, b_{s+4}$. Thus, $L(G^*(8, 1))$ admits a full set of dispersion free states.

$n \geq 10$. ($n$ even). In addition to the $\alpha_i$'s, define the $2n$ edge numberings on $G(n, 1)$ as follows. For $i = 0, 1, \ldots, n - 1$ let $\beta_i(a_i) = \beta_i(a_{i+3}) = \beta_i(a_{i+2k}) = \beta_i(b_i) = \beta_i(b_{i+3}) = \beta_i(b_{i+2k}) = \beta_i(c_{i+2}) = \beta_i(c_{i+5}) = 1$ where $k = 3, 4, \ldots, \frac{1}{2}(n - 2)$. Also let $\gamma_i(a_{i+2k}) = \gamma_i(b_{i+2k+1}) = 1$, where $k = 0, 1, \ldots, \frac{1}{2}(n - 2)$.

Condition 2 of Lemma 5 is satisfied by the $\beta_i$'s for $a_s \mathfrak{adj}\, a_t$, $b_s \mathfrak{adj}\, b_t$, and for $a_s \mathfrak{adj}\, b_t$ except when $t = s + 1$ and $t = s + n - 1$. It is satisfied by the $\gamma_i$'s for $a_s \mathfrak{adj}\, b_{s+1}$ and $a_s \mathfrak{adj}\, b_{s+n-1}$. Thus, $L(G^*(n, 1))$ admits a full set of dispersion free states when $n$ is even.

Conversely suppose $n$ is odd and that Conditions 1′ and 2′ of Lemma 5 are satisfied by an edge numbering $\alpha$. From Condition 1′ we have the $n$ equations $\alpha(a_i) + \alpha(c_{i+1}) + \alpha(a_{i+1}) = 1$ and the $n$ equations

$$\alpha(b_i) + \alpha(c_{i+1}) + \alpha(b_{i+1}) = 1 \ (i = 0, 1, \ldots, n - 1 \text{ and all}$$

subscripts are modulo $n$). From these equations it can be shown that

$$\alpha(a_0) - \alpha(a_{2k}) = \alpha(b_0) - \alpha(b_{2k}) \quad \text{and} \quad \alpha(a_0) + \alpha(a_{2k+1}) = \alpha(b_0) + \alpha(b_{2k+1}).$$

Since $n$ is odd, the last of these equations implies that $\alpha(a_0) = \alpha(b_0)$ so that together they imply that $\alpha(a_i) = \alpha(b_i)$ for all $i = 0, 1, \ldots, n - 1$.

From the above and Condition 2′, we have $\alpha(a_0) + \alpha(a_1) = \alpha(a_0) + \alpha(b_1) > 1$. However, $\alpha(a_0) + \alpha(c_1) + \alpha(a_1) = 1$ and $\alpha(c_1) \geq 0$ imply that $\alpha(a_0) + \alpha(a_1) \leq 1$. Thus, $L(G^*(n, 1))$ does not admit a full set of states if $n$ is odd.

One can see from the proof of Theorem 6 that the notation becomes a bit unwieldy. In view of the fact that there will be approximately 20 separate cases to consider for the theorem involving $k = 3$, we describe the edge numberings on $G(n, k)$ by means of $3 \times n$ matrices as follows. For each fixed $i = 0, 1, \ldots, n - 1$ the first row will consist of the edge numberings on $a_i, a_{i+1}, \ldots, a_{i+n-1}$ in that order. In like manner the $(j+1)$st element of the second row will be the numbering assigned to $b_{i+j}$ and the $(j+1)$st element of the third row will be the numbering assigned to $c_{i+j}$ for $j=0, 1, \ldots, n-1$. For example in $G(n, 1)$ with $n$ even and $n \geqslant 10$ (see Theorem 6), each $\alpha_i$ will be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \end{bmatrix}$$

Each $\beta_i$ will be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

Each $\gamma_i$ will be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

It might be worth noting when working with dispersion free states, that if the number of ones assigned to the outer rim ($a_j$'s) or to the inner rims ($b_j$'s) is less than $\frac{1}{2}n$ so that at least one of the spokes ($c_j$'s) gets a numbering of one, then if Condition 1 of Lemma 5 is satisfied it suffices to indicate the numberings on either the outer rims or the inner rims to uniquely determine the numberings for all other edges. However, it is hoped that the reader will appreciate the redundancy of listing all three rows of the matrices, for otherwise the other two would have to be calculated at every stage of the following proof. This way one only has to check that Condition 1 of Lemma 5 is satisfied. The reader is invited to do so.

**Theorem 7.** *Let $G(n, 2)$ be such that $L(G^*(n, 2))$ is an OMP. Then $L(G^*(n, 2))$ admits a full set of states if and only if $n \neq 5$. Furthermore, $L(G^*(n, 2))$ admits a full set of states but does not admit a full set of dispersion free states if and only if $n = 8$.*

**Proof.** We consider cases noting that smaller values of $n$ generally require a larger number of states.

$n = 3$ or 4. $1 < 2k < n$ is not satisfied.

$n = 5$. That $L(G^*(5, 2))$ does not admit a full set of states has already been observed.

$n = 6$. $L(G^*(6, 2))$ is not an OMP by Lemma 2.

Before proceeding further with individual cases, for each $n \geqslant 7$, we define $n$ edge numberings on $G(n, 2)$ as follows.

For each $i = 0, 1, \ldots, n-1$ define $\alpha_i$ by the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \end{bmatrix}.$$

It can be shown that if $a_s \cancel{\mathrm{adj}}\, c_t$, then there exists $\alpha_i$ such that $\alpha_i(a_s) = \alpha_i(c_t) = 1$. It can also be shown that if $b_s \cancel{\mathrm{adj}}\, c_t$, then there exists $\alpha_i$ such that $\alpha_i(b_s) = \alpha_i(c_t) = 1$ provided $t \neq s+1$. If $n \geq 9$ it is also true that there exists $\alpha_i$ such that $\alpha_i(c_s) = \alpha_i(c_t) = 1$. (Note that $c_s \cancel{\mathrm{adj}}\, c_t$ is always true for $s \neq t$.)

$n = 7$. In addition to the $\alpha_i$'s defined above, we define the 7 edge numberings $\beta_i$ on $G(7, 2)$ by the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally define the single edge numbering $\gamma(c_i) = 1$ and $\gamma(a_i) = \gamma(b_i) = 0$ for $i = 0, 1, \ldots, 6$.

Now note that $b_s \cancel{\mathrm{adj}}\, c_{s+1}$. But $\beta_s(b_s) = \beta_s(c_{s+1}) = 1$. If $a_s \cancel{\mathrm{adj}}\, a_t$, then there exists $\beta_i$ such that $\beta_i(a_s) = \beta_i(a_t) = 1$. If $b_s \cancel{\mathrm{adj}}\, b_t$, then there exists $\beta_i$ such that $\beta_i(b_s) = \beta_i(b_t) = 1$. Now $c_s \cancel{\mathrm{adj}}\, c_t$ for all $s \neq t$ but $\gamma(c_s) = \gamma(c_t) = 1$.

By Lemma 5, $L(G^*(7, 2))$ admits a full set of dispersion free states.

$n = 8$. It is easy to show that if $\alpha$ is any edge numbering on $G(8, 2)$ satisfying Conditions 1′ and 2′ of Lemma 5, then $1 < \alpha(a_0) + \alpha(a_3) \leq \frac{3}{2}$. This is not quite stringent enough to force the non-existence of a full set of states. However, if $\alpha(a_0) \in \{0, 1\}$ and $\alpha(a_3) \in \{0, 1\}$, then the above inequality is not satisfied. Thus $L(G^*(8, 2))$ does not admit a full set of dispersion free states. That it does admit a full set of states can be shown as follows.

Let the $\alpha_i$'s be defined as before. For each $i = 0, 1, \ldots, 7$, let $\beta_i$ be defined by the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\gamma(c_i) = 1$ and $\gamma(a_i) = \gamma(b_i) = 0$ for $i = 0, 1, \ldots, 7$.

It can be shown that the conditions of Lemma 5 are satisfied by the $\beta_i$'s if $b_s \cancel{\mathrm{adj}}\, b_t$, $a_s \cancel{\mathrm{adj}}\, b_t$ and $a_s \cancel{\mathrm{adj}}\, a_{s+2k}$, where $k = 1, 2, 3$. The $\gamma$'s take care of $c_s \cancel{\mathrm{adj}}\, c_t$. The cases $a_s \cancel{\mathrm{adj}}\, a_{s+3}$, $a_s \cancel{\mathrm{adj}}\, a_{s+5}$, and $b_s \cancel{\mathrm{adj}}\, c_{s+1}$ can be taken care of by defining the additional eight edge numberings as follows.

Let $i = 0, 1, \ldots, 7$. Define $\delta_i$ by

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Thus Conditions 1′ and 2′ of Lemma 5 are satisfied and $L(G^*(8,2))$ admits a full set of states.

$n=9$. For each $i=0,1,\ldots,8$ define $\beta_i$ by

$$\begin{bmatrix} 1&0&1&0&1&0&1&0&0\\ 1&1&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Also for each $i=0,1,\ldots,8$ define $\gamma_i$ by

$$\begin{bmatrix} 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0 \end{bmatrix}.$$

It can be seen that Condition 1 of Lemma 5 is satisfied by these 27 edge numberings (including the $\alpha_i$'s) and that Condition 2 is satisfied with the $\alpha_i$'s taking care of the cases indicated earlier; the $\beta_i$'s taking care of $a_s \operatorname{adj} a_t$, $b_s \operatorname{adj} b_t$, $a_s \operatorname{adj} b_t$; and the $\gamma_i$'s taking care of $b_s \operatorname{adj} c_{s+1}$.

$n=10$. For each $i=0,1,\ldots,9$ define $\beta_i$ by

$$\begin{bmatrix} 0&0&1&0&0&1&0&1&0&1\\ 1&0&0&1&0&0&1&1&0&0\\ 0&1&0&0&1&0&0&0&0&0 \end{bmatrix}.$$

For each $i=0,1,\ldots,9$ define $\gamma_i$ by

$$\begin{bmatrix} 1&0&1&0&0&1&0&1&0&0\\ 1&1&0&0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&1 \end{bmatrix}.$$

Then Conditions 1 and 2 of Lemma 5 hold with the $\alpha_i$'s taking care of the cases indicated earlier; the $\beta_i$'s taking care of $a_s \operatorname{adj} a_t$, $a_s \operatorname{adj} b_t$, $b_s \operatorname{adj} c_{s+1}$, $b_s \operatorname{adj} b_t$ except $t=s+5$; and the $\gamma_i$'s taking care of $b_s \operatorname{adj} b_{s+5}$.

$n=12$. For each $i=0,1,\ldots,11$ define $\beta_i$ by

$$\begin{bmatrix} 0&0&1&0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0&0&1&0 \end{bmatrix}.$$

For each $i=0,1,\ldots,11$ define $\gamma_i$ by

$$\begin{bmatrix} 1&0&1&0&1&0&1&0&1&0&1&0\\ 1&1&0&0&1&1&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Finally for each $i=0,1,\ldots,11$ define $\delta_i$ by

$$\begin{bmatrix} 1&0&1&0&0&1&0&1&0&0&0&0\\ 1&1&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&1&1&1 \end{bmatrix}.$$

It can be checked that Lemma 5 is satisfied with the $\alpha_i$'s used as before and the $\beta_i$'s taking care of $b_s \,\mathfrak{adj}\, b_{s+6}$, $a_s \,\mathfrak{adj}\, a_{s+3k}$, $b_s \,\mathfrak{adj}\, c_{s+1}$; the $\gamma_i$'s taking care of $a_s \,\mathfrak{adj}\, a_{s+2k}$, $b_s \,\mathfrak{adj}\, b_t$ except for $t = s+6$; and the $\delta_i$'s taking care of $a_s \,\mathfrak{adj}\, a_{s+5}$, $a_s \,\mathfrak{adj}\, a_{s+7}$.

$n = 13$. For each $i = 0, 1, \ldots, 12$ define $\beta_i$ by

$$\begin{bmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

For each $i = 0, 1, \ldots, 12$ define $\gamma_i$ by

$$\begin{bmatrix} 0&0&1&0&0&1&0&0&1&0&1&0&1 \\ 1&0&0&1&0&0&1&0&0&1&1&0&0 \\ 0&1&0&0&1&0&0&1&0&0&0&0&0 \end{bmatrix}.$$

The Conditions of Lemma 5 are satisfied with the $\alpha_i$'s used as before and the $\beta_i$'s taking care of $a_s \,\mathfrak{adj}\, b_t$; $b_s \,\mathfrak{adj}\, b_t$; $a_s \,\mathfrak{adj}\, b_t$; and the $\gamma_i$'s taking care of $b_s \,\mathfrak{adj}\, c_{s+1}$.

$n = 16$. For each $i = 0, 1, \ldots, 15$ define $\beta_i$ by

$$\begin{bmatrix} 0&0&1&0&0&1&0&0&1&0&0&1&0&1&0&1 \\ 1&0&0&1&0&0&1&0&0&1&0&0&1&1&0&0 \\ 0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0 \end{bmatrix}.$$

For each $i = 0, 1, \ldots, 15$ define $\gamma_i$ by

$$\begin{bmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

It can be seen that the conditions of Lemma 5 hold with $\alpha_i$'s used as before and the $\beta_i$'s taking care of $a_s \,\mathfrak{adj}\, b_t$, $b_s \,\mathfrak{adj}\, b_t$, $b_s \,\mathfrak{adj}\, c_{s+1}$, $b_s \,\mathfrak{adj}\, b_t$ except for $t = s + 3k + 2$ with $k = 1, 2, 3$; and the $\gamma_i$'s taking car of $b_s \,\mathfrak{adj}\, b_{s+3k+2}$ with $k = 1, 2, 3$.

Because they play a special role in the definition of the $\beta_i$'s for the general cases, we will define the following submatrices:

$$A = \begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0&1&0&1 \\ 1&1&0&0 \\ 0&0&0&0 \end{bmatrix}.$$

Now let $n = 4p + 3$ for $p \geq 2$. For $i = 0, 1, \ldots, n-1$ define $\beta_i$ by the matrix $(A\,B\,B \ldots B)$, where $B$ appears $p$ times.

Let $n = 4p + 2$ for $p \geq 3$. For $i = 0, 1, \ldots, n-1$ define $\beta_i$ by $(A\,A\,B\,B \ldots B)$, where $B$ appears $p - 1$ times.

Let $n = 4p + 1$ for $p \geq 4$. For $i = 0, 1, \ldots, n-1$ define $\beta_i$ by $(A\,A\,A\,B\,B \ldots B)$, where $B$ appears $p-2$ times.

Let $n = 4p$ for $p \geq 5$. For $i = 0, 1, \ldots, n-1$ define $\beta_i$ by $(A\,A\,A\,A\,B\,B \ldots B)$, where $B$ appears $p-3$ times.

For the last four general cases, it can be seen that the conditions of Lemma 5 are satisfied with the $\alpha_i$'s taking care of the cases indicated earlier and the $\beta_i$'s taking care of $a_s \not\mathrm{adj}\, a_t$, $b_s \not\mathrm{adj}\, b_t$, $a_s \not\mathrm{adj}\, b_t$ and $b_s \not\mathrm{adj}\, c_{s+1}$. This proves the theorem. □

**Theorem 8.** *Let $G(n, 3)$ be such that $L(G^*(n, 3))$ is an OMP. Then $L(G^*(n, 3))$ admits a full set of dispersion free states for all $n$.*

**Proof.** First note that $n \geq 7$ since $1 < 2k < n$.

Before proceeding with individual cases, for each $n \geq 7$ we define $n$ edge numberings on $G(n, 3)$ by the matrices described above as follows.

For each $i = 0, 1, \ldots, n-1$ define $\alpha_i$ by the matrix

$$\begin{bmatrix} 1\;0\;0\;1\;0\;0\;0\;0\;0 \ldots 0 \\ 1\;1\;0\;0\;0\;0\;0\;0\;0 \ldots 0 \\ 0\;0\;1\;0\;0\;1\;1\;1\;1 \ldots 1 \end{bmatrix}.$$

It can be shown that if $b_s \not\mathrm{adj}\, c_t$, then there exists $\alpha_i$ such that $\alpha_i(b_s) = \alpha_i(c_t) = 1$. If $n \geq 8$ it can be shown that there exists $\alpha_i$ such that $\alpha_i(a_s) = \alpha_i(c_t) = 1$ whenever $a_s \not\mathrm{adj}\, c_t$ and that there exists $\alpha_i$ such that $\alpha_i(c_s) = \alpha_i(c_t) = 1$. (Note it is always true that $c_s \not\mathrm{adj}\, c_t$.)

We now consider individual cases in ascending order. It may be well to note that some values of $n$ appear to be absent as we progress through the proof. These are merely part of the general cases which will be considered near the end of the proof.

$n = 7$. In addition to the $\alpha_i$'s defined above define the $n$ edge numberings $\beta_i$ by the matrix

$$\begin{bmatrix} 1\;0\;1\;0\;1\;0\;0 \\ 1\;1\;1\;0\;0\;0\;0 \\ 0\;0\;0\;0\;0\;0\;1 \end{bmatrix}.$$

Also define the single edge numbering $\gamma$ by $\gamma(c_k) = 1$ for $k = 0, 1, \ldots, n-1$.

Condition 2 of Lemma 5 is satisfied by the $\alpha_i$'s for $a_s \not\mathrm{adj}\, c_t$ unless $t = s + 4$, and for $b_s \not\mathrm{adj}\, c_t$ as indicated above. Condition 2 is satisfied by the $\beta_i$'s for $a_s \not\mathrm{adj}\, a_t$, for $a_s \not\mathrm{adj}\, b_t$, for $b_s \not\mathrm{adj}\, b_t$, and for $a_s \not\mathrm{adj}\, c_{s+4}$. Finally, $\gamma$ takes care of $c_s \not\mathrm{adj}\, c_t$ since every $c_k$ has edge numbering one. By Lemma 5, $L(G^*(7, 3))$ admits a full set of dispersion free states.

$n=8$. In addition to the $\alpha_i$'s define the $n$ edge numberings $\beta_i$ by

$$\begin{bmatrix} 1&0&1&0&1&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&1 \end{bmatrix},$$

and the $n$ edge numberings $\gamma_i$ by

$$\begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Condition 2 of Lemma 5 is satisfied by the $\beta_i$'s for $a_s \mathfrak{adj}\, a_t$ unless $t=s+3$ or $t=s+5$, for $b_s \mathfrak{adj}\, b_t$ unless $t=s+4$, and for $a_s \mathfrak{adj}\, b_t$ unless $t=s+3$. Condition 2 is satisfied by the $\alpha_i$'s for $a_s \mathfrak{adj}\, a_{s+3}$ and $a_s \mathfrak{adj}\, a_{s+5}$ and by the $\gamma_i$'s for $b_s \mathfrak{adj}\, b_{s+4}$ and $a_s \mathfrak{adj}\, b_{s+3}$. Thus, $L(G^*(8,3))$ admits a full set of dispersion free states. (Recall that $a_s \mathfrak{adj}\, c_t$, $b_s \mathfrak{adj}\, c_t$, and $c_s \mathfrak{adj}\, c_t$ are taken care of by the $\alpha_i$'s as indicated earler.)

$n=9$. $L(G^*(9,3))$ is not an OMP by Lemma 2.

$n=10$. Define the $n$ edge numberings $\beta_i$ by

$$\begin{bmatrix} 0&0&1&0&1&0&1&0&0&1\\ 1&0&1&0&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&1&0 \end{bmatrix},$$

the $n$ edge numberings $\gamma_i$ by

$$\begin{bmatrix} 1&0&1&0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&1&1 \end{bmatrix},$$

and the $n$ edge numberings $\delta_i$ by

$$\begin{bmatrix} 1&0&0&1&0&1&0&0&1&0\\ 1&1&0&0&0&1&1&0&0&0\\ 0&0&1&0&0&0&0&1&0&0 \end{bmatrix}.$$

Condition 2 of Lemma 5 is satisfied by the $\beta_i$'s for $a_s \mathfrak{adj}\, a_t$; for $b_s \mathfrak{adj}\, b_t$ unless $t=s+1$, $t=s+5$, or $t=s+9$; and for $a_s \mathfrak{adj}\, b_t$ unless $t=s+9$. Condition 2 is satisfied by the $\gamma_i$'s for $b_s \mathfrak{adj}\, b_{s+1}$, for $b_s \mathfrak{adj}\, b_{s+9}$, and for $a_s \mathfrak{adj}\, b_{s+9}$. Finally, Condition 2 is satisfied by the $\delta_i$'s for $b_s \mathfrak{adj}\, b_{s+5}$. Thus, $L(G^*(10,3))$ admits a full set of dispersion free states.

It will now be observed that from this point on, each of the $\beta_i$'s defined will satisfy Condition 2 of Lemma 5 for $a_s \mathfrak{adj}\, a_t$, for $b_s \mathfrak{adj}\, b_t$, and for $a_s \mathfrak{adj}\, b_t$ except for the case $n=16$ (although they will still have to be defined differently from case to case until we finally reach the general cases). Thus, we will merely define the $\beta_i$'s for each of the remaining cases and leave it up to the reader to check that Condition 2 is satisfied by the $\alpha_i$'s and $\beta_i$'s as indicated.

Because they play a special role in the definition of the $\beta_i$'s, we will define the following submatrices.

$$A=\begin{bmatrix}1&0&0&1&0\\1&1&0&0&0\\0&0&1&0&0\end{bmatrix},\qquad B=\begin{bmatrix}1&0&1&0&1&0\\1&1&1&0&0&0\\0&0&0&0&0&0\end{bmatrix},\quad\text{and}\quad C_k=\begin{bmatrix}0&0&0&\dots&0\\0&0&0&\dots&0\\1&1&1&\dots&1\end{bmatrix},$$

where each row of $C_k$ consists of exactly $k$ entries.

$n=11$. Define the $\beta_i$'s by $(A\,B)$.

$n=12$. Define the $\beta_i$'s by $(A\,B\,C_1)$.

$n=13$. Define the $\beta_i$'s by $(A\,B\,C_2)$.

$n=14$. Define the $\beta_i$'s by $(A\,B\,C_3)$.

$n=15$. Define the $\beta_i$'s by $(A\,B\,C_4)$.

$n=16$. Define the $\beta_i$'s by $(A\,A\,B)$ and the $\gamma_i$'s by

$$\begin{bmatrix}1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}.$$

This time we notice that the $\beta_i$'s do not satisfy Condition 2 of Lemma 5 for $b_s$ and $b_{s+8}$. This problem is taken care of by the $\gamma_i$'s. (It might be worth noting that it does not work to define the $\beta_i$'s by $(A\,B\,C_5)$.

$n=18$. Define the $\beta_i$'s by $(A\,A\,B\,C_2)$.

$n=19$. Define the $\beta_i$'s by $(A\,B\,B\,C_2)$.

$n=20$. Define the $\beta_i$'s by $(A\,B\,B\,C_3)$.

$n=21$. Define the $\beta_i$'s by $(A\,B\,B\,C_4)$.

$n=24$. Define the $\beta_i$'s by $(A\,A\,B\,B\,C_2)$.

$n=25$. Define the $\beta_i$'s by $(A\,A\,B\,B\,C_3)$.

$n=26$. Define the $\beta_i$'s by $(A\,A\,B\,B\,C_4)$.

$n=30$. Define the $\beta_i$'s by $(A\,A\,B\,B\,B\,C_2)$.

$n=31$. Define the $\beta_i$'s by $(A\,A\,B\,B\,B\,C_3)$.

$n=36$. Define the $\beta_i$'s by $(A\,A\,B\,B\,B\,B\,C_2)$.

$n=6p+5$, for $p\geqslant 2$. Define the $\beta_i$'s by $(A\,B\,B\dots B)$, where $B$ appears $p$ times.

$n=6p+4$ for $p\geqslant 3$. Define the $\beta_i$'s by $(A\,A\,B\,B\dots B)$, where $B$ appears $p-1$ times.

$n=6p+3$ for $p\geqslant 4$. Define the $\beta_i$'s by $(A\,A\,A\,B\,B\dots B)$, where $B$ appears $p-2$ times.

$n=6p+2$ for $p\geqslant 5$. Define the $\beta_i$'s by $(A\,A\,A\,A\,B\,B\dots B)$, where $B$ appears $p-3$ times.

$n=6p+1$ for $p\geqslant 6$. Define the $\beta_i$'s by $(A\,A\,A\,A\,A\,B\,B\dots B)$, where $B$ appears $p-4$ times.

$n = 6p$ for $p \geqslant 7$. Define the $\beta_i$'s by $(A\,A\,A\,A\,A\,A\,B\,B \ldots B)$, where $B$ appears $p-5$ times.

This takes care of all possible cases and the theorem is proved. □

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